Journal of Statistical Physics, Vol. 70, No. 3/4, 1993

Resonant Response of a Thermalized Ensemble of Nonlinear Oscillators

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Received April 7, 1992

The analysis is carried out of the response of the center of gravity (dipole moment) of the distribution of noninteracting thermalized nonlinear oscillators to a sinusoidal driving force. Heat bath coupling is modeled by damping and noise. The driving is weak, but the frequency is resonant, so that there is a nonlinear resonance in the phase space. The response has a linear part that can be obtained from the perturbation analysis and a small nonlinear correction that is specific for the resonant structure.

KEY WORDS: Nonlinear resonance; distribution function; Fokker–Planck equation.

1. INTRODUCTION

The response functions of linear and nonlinear Hamiltonian oscillators are conventionally studied and applied in nonlinear mechanics, circuit theory, and other fields. The influence of a random environment often plays an important role and is studied in a variety of problems as well. In the present paper, we consider one problem of that kind of an apparently generic nature: the steady-state response of a thermalized ensemble of non-interacting nonlinear oscillators to a sinusoidal driving. "Thermalized" here implies the interaction with the heat bath, modeled by damping and noise.

The steady-state response is defined as the average (over the ensemble) time-dependent displacement of particles from their equilibrium in the large-time limit, when all transients have died out. We are specifically interested in the "resonant" situation when the frequency of the driving falls

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0022-4715/93/0200-0939$07.00/0

within the core of the thermal distribution of frequencies of the oscillators. The driving, assumed to be small, creates an (isolated) nonlinear resonance at the driving frequency, and the emphasis is on finding the impact of this resonance on the response.

The results and implications of our analysis bear many analogies with the theory of the Landau damping of wave propagation in collisionless plasmas.[1] The dynamics of particles in the potential of these waves is essentially the same as what one has near a nonlinear resonance. One example of such parallelism is the Landau damping of bunched beam modes in the theory of coherent instabilities of particle beams in electron and proton storage rings.[2] Our analysis is qualitatively different in that it applies to the long-time limit, when the nonlinear trapping of particles in the resonance (wave bucket in plasma terminology), implicitly neglected in Landau damping theory, is essential.

2. THE MODEL

Consider a distribution of noninteracting particles in the nonlinear potential, coupled to the heat bath as modeled by damping and noise and subject to periodic driving:

$$\ddot{x} + \omega^2 x + \lambda x^3 = \varepsilon \sin \Omega t - \alpha \dot{x} + (2\eta)^{1/2} \xi(t) \tag{1}$$

Here ω is the linear frequency, $\lambda > 0$ is the amplitude of the nonlinearity, $\varepsilon > 0$ is the driving amplitude, Ω is the driving frequency, α is the damping decrement, η is the noise amplitude, and $\xi(t)$ is the white noise random process $\langle \xi(t) \xi(t+\tau) \rangle = \delta(\tau)$. We will assume the coupling to the heat bath to be very weak, $\alpha \sim \eta \ll \omega, \lambda, \varepsilon$, so that the longest time scale in the problem is the relaxation time $\tau_r = 1/\alpha$. The driving amplitude ε will be assumed to be small, $\varepsilon \ll (\lambda/\omega^3)(\eta/\alpha)$, to allow the description of Hamiltonian dynamics of (1) ($\alpha = 0$, $\eta = 0$) in terms of isolated nonlinear resonances. The last, and least important, assumption will be the smallness of the nonlinearity $\lambda \ll \omega^4(\alpha/\eta)$, which is just made in order to avoid unessential complications. The extension to an arbitrary λ, and, for that matter, to an arbitrary nonlinear potential is straightforward.

We will be analyzing the response of the centroid motion:

$$D(t) = \int x\rho(x, p, t)\, dx\, dp \tag{2}$$

to the weak perturbation in the "resonant" case, by which we will understand in this context that the driving frequency Ω falls within the core of the distribution of frequencies of the oscillators. This latter condition can be shown to be:

$$\frac{\lambda\eta}{\alpha\omega^2} \gtrsim \Omega - \omega > 0 \tag{3}$$

The evolution of the distribution of particles in the stochastic system (1) is determined by the Fokker–Planck equation (FPE)[3]:

$$\frac{\partial\rho}{\partial t} + p\frac{\partial\rho}{\partial x} - (\varepsilon \sin \Omega t + \omega^2 x + \lambda x^3)\frac{\partial\rho}{\partial p} = \frac{\partial}{\partial p}\left(\alpha p\rho + \eta\frac{\partial\rho}{\partial p}\right) \tag{4}$$

In the absence of perturbation, $\varepsilon = 0$, the stationary solution is the canonical distribution:

$$\rho_0 = \frac{1}{N_0}\exp\left[-\frac{\alpha}{\eta}H_0(x, p)\right] \tag{5}$$

where $N_0 = 2\pi(\eta/\omega\alpha)$ is the normalization factor and H_0 is the unperturbed Hamiltonian:

$$H_0 = \frac{p^2}{2} + \omega^2\frac{x^2}{2} + \lambda\frac{x^4}{4} \tag{6}$$

Consider first the perturbed motion $\varepsilon \neq 0$ in the absence of damping and noise. The perturbed Hamiltonian is

$$H = H_0(x, p) + \varepsilon x \sin \Omega t \tag{7}$$

We will describe the nonlinear-resonant motion in the system (7) following the conventional approach.[4,5] For that, we have to transform first to the action–angle variables of the unperturbed Hamiltonian (6). This can be done exactly, but to avoid cumbersome manipulations with special functions we will rather take advantage of the smallness of the nonlinearity λ. The action–angle variables then will be (in the lowest approximation in λ) just that of the linear motion

$$I = \frac{1}{\omega}\left(\frac{p^2}{2} + \omega^2\frac{x^2}{2}\right), \qquad \theta = \arctan\left(\frac{p}{x\omega}\right)$$

and the Hamiltonian H_0 should be averaged over θ, yielding

$$H_0 = \omega I + \frac{3}{8}\frac{\lambda}{\omega^2}I^2 \tag{8}$$

We introduce now the resonant phase $\Psi = \theta - \Omega t$ and present the perturbed Hamiltonian as

$$H = H_0 - \Omega I + \varepsilon \left(\frac{I}{2\omega}\right)^{1/2} [\sin(\Psi + 2\Omega t) - \sin \Psi] \tag{9}$$

In the vicinity $|I - I_r| \sim \sqrt{\varepsilon}$ of the resonance $\Omega = \nu(I_r)$ [where $\nu(I)$ is the frequency of the unperturbed oscillations, $\nu(I) = dH_0/dI$] one can keep only the time-independent (resonant) harmonic, expand the Hamiltonian $H_0(I)$ up to quadratic terms in the deviation $q = I - I_r$ of the action I from the resonant value I_r, drop the nonresonant harmonic, and put the amplitude of the sin Ψ term to its value at $I = I_r$, yielding the pendulum Hamiltonian:

$$H_r = k\frac{q^2}{2} - V \sin \Psi \tag{10}$$

where $k = d\nu(I_r)/dI$ and $V = \varepsilon(I_r/2\omega)^{1/2}$.

3. THE STEADY-STATE DISTRIBUTION

Our goal now is to find the steady-state distribution function that establishes itself in the limit of long time $t \gg \tau_r = 1/\alpha$, and compute the steady-state response $D(t)$ (2), from it. Since the parameter ε (and respectively V) was assumed to be small, we will be limiting ourselves to the approximate analysis of distribution function in powers of $V^{1/2}$, operating with the precision that allows one to produce only two terms in the response: linear in V and the leading-order nonlinear correction. The methods of calculation will be different in the resonant region $|q| \sim (V/|k|)^{1/2}$ and in the nonresonant one. Consider first the former one.

The smallness of damping and noise in the model was introduced in order to make use of a relatively well-developed technique of the calculation of the steady-state distribution in that limit, when it becomes constant along the Hamiltonian trajectories.[6-8] The method is the elimination of "fast" variables in the FPE by averaging along the Hamiltonian trajectories. For our specific system in the resonant region, it is convinient to carry out this averaging in two steps, the first being the introduction of the resonant-action variables Ψ, I and averaging over (nonresonant) phase θ, and the second the averaging along the pendulum trajectories (10) in the resonant variables (ref. 9, pp. 92, 101). As a result of the first averaging, the FPE for the steady-state distribution takes the form

$$\frac{\partial H_r}{\partial I}\frac{\partial \rho}{\partial \Psi} - \frac{\partial H_r}{\partial \Psi}\frac{\partial \rho}{\partial I} = \frac{\partial}{\partial I}\left(\alpha I \rho + \eta \frac{I}{\omega}\frac{\partial \rho}{\partial I}\right) \tag{11}$$

with the pendulum Hamiltonian H_r, (10). The second averaging in the FPE (11) yields the o.d.e. for the H-only dependent distribution, which can be solved explicitly. While the detailed derivation is available in the original references,[7,8] the sketch is presented in Appendix for the sake of self-consistency. The distribution can be represented by two different expressions inside and outside the separatrix. The former [$sH_r < V$, $s = \text{sgn}(k)$] is [see (A7)]

$$\rho_{\text{in}} = \frac{1}{N} \exp\left(-\frac{\alpha}{\eta} \omega \frac{k(q^2/2) - V \sin \Psi}{kI_r} \right) \tag{12}$$

where N is the normalization constant. Outside the separatrix ($sH_r > V$) one obtains [see (A9)]

$$\rho'_{\pm}(H_r) = \frac{1}{N} \exp\left\{ -\frac{\alpha}{\eta} \omega \left[\frac{V}{|k| I_r} \pm \left(\frac{V}{|k|}\right)^{1/2} G\left(\frac{sH_r}{V}\right) \right] \right\} \tag{13}$$

$$G(y) = \frac{\pi}{2\sqrt{2}} \int_1^y \frac{dz}{(1+z)^{1/2} E([2/(1+z)]^{1/2})} \tag{14}$$

where + and − refer, respectively, to the region above ($q > 0$) and below ($q < 0$) the separatrix, and $E(y)$ is the complete elliptic integral of the second kind.

Expressions (12) and (13) define the distribution function in the small region of the vicinity of the resonance $|q| \ll I_r$ where the pendulum approximation (10) is applicable. In the nonresonant region $|q| \sim I_r$, the distribution function can be found through the perturbative analysis of the same Liouville equation (15). The lowest-order approximation ρ_0 in the nonresonant region has obviously the same unperturbed form as (5), $\rho_0 = (1/N_1) \exp[-(\alpha/\eta)\omega I]$. The normalization factor N_1 equals N_0 only in the lowest order, while the correction $N_1 - N_0 \sim \sqrt{V}$ appears due to the presence of the resonance and has to be found separately. The factor N_1 is different above ($q > 0$) and below ($q < 0$) the resonance and can be found from the asymptotics $sH_r/V \gg 1$ of the resonant distribution (13). Using the asymptotics of the function $G(y)$ of (14), $G(y) \xrightarrow[y \gg 1]{} (2y)^{1/2} + C_1$ with the numerical constant

$$C_1 = -\sqrt{2} + \frac{\pi}{2\sqrt{2}} \int_1^\infty dz \left[\frac{1}{(1+z)^{1/2} E([2/(1+z)]^{1/2})} - \frac{2}{\pi\sqrt{z}} \right] \tag{15}$$

one obtains the nonresonant distribution up to the terms $\sim \sqrt{V}$ as

$$\rho_{0\pm} = \frac{1}{N} \left[1 \mp \frac{\alpha}{\eta} \omega \left(\frac{V}{|k|}\right)^{1/2} C_1 \right] \exp\left(-\frac{\alpha}{\eta} \omega I \right) \tag{16}$$

The normalization factor N can be calculated to the necessary precision of $\sim V^{1/2}$ by integrating the distribution (16) over I, yielding

$$N = N_0 \exp\left(\frac{\alpha}{\eta}\omega I_r\right)\left\{1 + \frac{\alpha}{\eta}\omega\left(\frac{V}{k}\right)^{1/2} C_1\left[1 - 2\exp\left(-\frac{\alpha}{\eta}\omega I_r\right)\right]\right\} \quad (17)$$

The perturbative correction ρ_1 up to the terms $\sim V^{3/2}$ can be obtained from the linearized Vlasov equation (dropping the "thermal" $\sim\alpha, \eta$ terms in the FPE):

$$\frac{\partial\rho_1}{\partial t} + \nu(I)\frac{\partial\rho_1}{\partial I} - \varepsilon\left(\frac{I}{2\omega}\right)^{1/2}[\cos(\theta+\omega t) - \cos(\theta-\omega t)]\frac{\partial\rho_{0\pm}}{\partial I} = 0 \quad (18)$$

The result is

$$\rho_{1\pm} = \frac{1}{N}\left[1 \pm \frac{\alpha}{\eta}\omega\left(\frac{V}{|k|}\right)^{1/2} C_1\right]\frac{\alpha\varepsilon(I\omega/2)^{1/2}}{\eta}\exp\left(-\frac{\alpha}{\eta}\omega I\right) \times\left[\frac{\sin(\theta+\Omega t)}{\nu(I)+\Omega} - \frac{\sin(\theta-\Omega t)}{\nu(I)-\Omega}\right] \quad (19)$$

Notice here that the standard linearization procedure with the unperturbed distribution ρ_0 in (5) would give the same expression (19) with N_0 instead of N and without the first term in brackets, thus giving a correct linear-in-V part but not the $\sim V^{3/2}$ one. The range of applicability of the perturbative solution $\rho_\pm = \rho_{0\pm} + \rho_{1\pm}$ can be easily seen to be $|q| \gg (V/|k|)^{1/2}$, thus overlapping with the range of applicability of the resonant distribution (13).

4. THE STEADY-STATE RESPONSE

Having the steady-state distribution defined by the expressions (12)–(19), we are in a position now to calculate the response $D(t)$ of (2). First, changing the variables of integration from x, p to I, θ, one immediately observes that all expressions (12), (13), (16), and (19) for the distribution function yield the response in the same phase, so that

$$D(t) = A\sin\Omega t \quad (20)$$

Thus, there is no phase rotation between the driving and the response, which is a natural consequence of the smallness of the damping α.

The response amplitude A has to be calculated by integration of (2) over all the phase space. Splitting the integral into two separate contributions from the "resonant" region $|q| < q_c$ [where $q_c = R(V/|k|)^{1/2}$ with the constant $R \gg 1$] and from the "nonresonant" one $|q| > q_c$, one can calculate the former using the resonant distribution (12), (13), and the latter using the perturbative distribution $\rho_\pm = \rho_{0\pm} + \rho_{1\pm}$. The integration of the distribution (12), (13) over the range $|q| < q_c$ can be seen, using the antisymmatry of the argument of the exponential in expression (13) relative to q, to give the contribution $\sim V^{3/2}$ for any finite R. The smallness of this contribution, however, exceeds the precision of the derivation of the distribution (12), (13), since the next-order correction to the distribution (13) [coming from the dropped terms in the transition from the full Hamiltonian (9) to the pendulum approximation (10)] can be estimated to yield a contribution of the same order of magnitude. Realizing as well that the consistent calculation of such a correction is not a straightforward extension of the presented approach, we choose instead to drop all the contributions $\sim V^{3/2}$ to the response A altogether and to keep only the terms of the lower order. Those have to be found therefore only in the nonresonant contribution from the region $|q| > q_c$, which after the trivial integration over θ presents itself as

$$A = 2\varepsilon \frac{\pi}{N} \frac{\alpha}{\eta} \int_{|q| > q_c} I \exp\left(-\frac{\alpha}{\eta}\omega q\right) v(I) \left[\frac{1}{v^2(I) - \Omega^2} + \frac{s(\alpha/\eta)\,\omega(V/k)^{1/2} C_1}{|v^2(I) - \Omega^2|}\right] dI \tag{21}$$

The lowest-order linear-in-V part of this integral has the form familiar from plasma theory[1] with the principal-value integration:

$$A_0 = 2\varepsilon \frac{\pi}{N_0} \frac{\alpha}{\eta} \int_{\text{P.V.}} I \exp\left(-\frac{\alpha}{\eta}\omega q\right) \frac{v(I)}{v^2(I) - \Omega^2}\, dI \tag{22}$$

The second term in formula (21) will produce the only nonlinear contribution of order lower than $V^{3/2}$. This dominating nonlinear contribution can be found by putting $N = N_0$, taking a finite but large value of R, and keeping only the logarithmically dominating part, yielding

$$A_1 = \frac{\varepsilon^{3/2}}{2} \left(\frac{\alpha}{\eta}\right)^2 \frac{\omega^2 I_r}{k\sqrt{|k|}} \left(\frac{I_r}{2\omega}\right)^{1/4} C_1 \ln\left(\frac{I_r |k|^{1/2} (2\omega/I_r)^{1/4}}{\sqrt{\varepsilon}}\right) \tag{23}$$

It should be noticed here that retaining only the logarithmically-dominating term does not allow one to find the numerator under the logarithm, and the one in formula (23) is but an arbitrary choice. C_1 is a numerical constant, (15).

5. CONCLUSIONS

Our final result for the response A consists of two expressions: formula (22) for the lowest-order linear-in-ε part A_0 and formula (23) for the first nonlinear correction A_1 of the order $\varepsilon^{3/2}\ln(1/\varepsilon)$. It is important to realize that this order of magnitude of nonlinear correction is due to the influence of the nonlinear resonance. In the nonresonant case $\Omega < \omega$ the lowest-order part of the response $\sim\varepsilon$ would be given by the same formula (22), while the first nonlinear correction would emerge from a conventional perturbation analysis to be of the order of ε^2. It is also interesting to notice that unlike the linear part A_0 of (22), the nonlinear correction A_1 of (23) is associated only with the particles in the close vicinity of the resonance $I = I_r$, which is manifested in the dependence on the function $\nu(I)$ only through its local behavior at the point $I = I_r$.

A general remark relates to an unusual logarithmic order of the correction A_1. Basically, it can attributed to a "nonlocality" of the influence of the resonance on the steady-state distribution: though the strong effect of the perturbation $\sim\varepsilon^{1/2}$ on the trajectories is limited to the region $|I - I_r| \sim \varepsilon^{1/2}$, the steady-state distribution is perturbed by $\sim\varepsilon^{1/2}$ even far from the resonance [see formula (16)]. This nonlocality becomes possible in view of an arbitrarily large time that is allowed for the relaxation to the steady state, so that even a small damping and noise redistribute the particles according to the "global" balance of density. Most graphically, this argument manifests itself in comparison with the steady-state response of the same ensemble of nonlinear oscillators without damping and noise. Preliminary considerations in the model with either adiabatic or instantaneous turn-on of the perturbation indicate that the leading nonlinear correction to the response is of the order $\varepsilon^{3/2}$ and the logarithmic term is absent.

APPENDIX

In this Appendix, we briefly introduce the method of solving the FPE (11) in the limit $\alpha, \eta \to 0$, following refs. 7 and 8. In that limit, the solution ρ depends on phase-space variables I, Ψ only through the combination $H_r(I, \Psi)$. The equation for this solution is obtained by plugging it into the FPE (11) (lhs yields zero) and averaging all emerging functions of I and Ψ over time while assuming that the time dependence of $I(H_r, t)$ and $\Psi(H_r, t)$ is like that of the Hamiltonian motion with the Hamiltonian H_r of (10). The most useful flux conservation form is obtained in the action variable $J(H_r)$ of the pendulum (10),

$$\frac{\partial}{\partial J}\left(\alpha F(J)\rho + \frac{\eta}{\omega} D(J)\frac{\partial\rho}{\partial J}\right) = 0 \tag{A1}$$

where the quantities F and D are defined by

$$F(J) = \left\langle I \frac{\partial J}{\partial I} \right\rangle \tag{A2}$$

$$D(J) = \left\langle I \left(\frac{\partial J}{\partial I} \right)^2 \right\rangle \tag{A3}$$

The partial derivatives here are for $\Psi = \text{const}$ and the angle brackets stand for the above-described averaging over time.

One can easily obtain now the zero-flux solution of Eq. (A1) by straightforward integration (the flux is the term under the external differentiation). Returning to the variable $H_r(J)$ in the resulting integral yields

$$\rho = \exp\left(-\frac{\alpha}{\eta} \omega \int dH_r \frac{F(H_r)}{D(H_r)} \right) \tag{A4}$$

We will consider the distribution function only in the vicinity of the nonlinear resonance $|I - I_r| = |q| \ll \eta/\alpha\omega$, where we can compute the averages in the leading order in q (or H_r) to be

$$F = \begin{cases} k\langle q^2 \rangle & \text{if} \quad sH_r < V \\ kI_r\langle q \rangle & \text{if} \quad sH_r > V \end{cases} \tag{A5}$$

$$D = k^2 I_r \langle q^2 \rangle \tag{A6}$$

where $s = \text{sgn}(k)$. Note here that the expressions for F are quite different for the region inside the separatrix, $sH_r < V$ (where $\langle q \rangle = 0$) and the region outside the separatrix, $sH_r > V$. Thus, we obtain an explicit formula for the region inside the separatrix ($sH_r < V$)

$$\rho_{\text{in}} = \frac{1}{N} \exp\left(-\frac{\alpha}{\eta} \omega \frac{(k(q^2/2) + V \sin \Psi)}{kI_r} \right) \tag{A7}$$

and two different expressions for ρ_+ in the region above the separatrix, $sH_r > V$, $q > 0$, and for ρ_- in the region below the separatrix, $sH_r > V$, $q < 0$:

$$\rho_\pm = \frac{1}{N} \exp\left[-\frac{\alpha}{\eta} \omega \left(\frac{V}{kI_r} \pm \frac{1}{k} \int_V^{H_r} dH_1 \frac{\langle q \rangle}{\langle q^2 \rangle} \right) \right] \tag{A8}$$

Here we also used the condition of the continuity of the distribution function at the separatrix $sH_r = V$.

The two averages in the expression (A8) can be calculated by changing the variable of integration in the definition of the average $\langle \cdots \rangle =$

$1/T(H_r) \int_0^T dt...$, with $T(H_r)$ being the period of oscillation of the pendulum (10), from t to Ψ through $dt = d\Psi/\dot{\Psi} = d\Psi/kq(H_r, \Psi)$ (the phase-space variables have to be expressed then as functions of H_r and Ψ), yielding

$$\rho_{\pm}(H_r) = \frac{1}{N} \exp\left\{ -\frac{\alpha}{\eta}\omega \left[\frac{V}{kI_r} \pm \left(\frac{V}{k}\right)^{1/2} G\left(\frac{sH_r}{V}\right) \right] \right\} \tag{A9}$$

$$G(y) = \frac{\pi}{2\sqrt{2}} \int_1^y \frac{dz}{(1+z)^{1/2} E([2/1+z)]^{1/2})} \tag{A10}$$

where $E(y)$ is the complete elliptic integral.

REFERENCES

1. F. Chen, *Introduction to Plasma Physics* (Plenum Press, New York, 1974), p. 213.
2. A. Chao, Coherent instabilities of a relativistic bunched beams, in *Proceedings 2nd Summer School on High Energy Accelerators, SLAC, 1982* (SLAC-PUB-2946, 1982).
3. H. Risken, The *Fokker–Planck Equation* (Springer, Berlin, 1983).
4. B. Chirikov, *Phys. Rep.* **52**:263 (1979).
5. A. Lichtenberg and M. Lieberman, *Regular and Stochastic Motion* (Springer, Berlin, 1983).
6. K. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1985).
7. S. Zatsepin, *Teor. Mat. Fiz.* **1**:55 (1983).
8. J. Schonfeld, *Ann. Phys.* **160**:149 (1985).
9. A. Gerasimov, *Physica D* **41**:89 (1990).